On the Aberration of Sloped Lenses and on their Adaptation to Telescopes of Unequal Magnifying Power in Perpendicular Directions.

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The present paper consists of two parts to a large extent independent. The reader who does not care to follow the details of calculation may take the results relative to unsymmetrical aberration for granted. The subject of the second part is somewhat larger than the title. It treats of the advantage which often attends a magnification unequal in different directions and of the methods available for attaining it. Among these is the method of the sloped object-lens. Such sloping introduces in general unsymmetrical aberration. The intention of the first part is largely to show how this may be minimised so as to become unimportant.

Part I.

Before proceeding to actual calculations of the aberrations of a sloped lens, it may be well to consider briefly the general character of a pencil of rays affected with unsymmetrical aberration.

The axis of the pencil being taken as axis of z, let the equation of the wave-surface, to which all rays are normal, be

$$z = \frac{x^2}{2\rho}+\frac{y^2}{2\rho'}+\alpha x^3+\beta x^2 y+\gamma x y^2+\delta y^3+\dots \tag{1}$$

The principal focal lengths, measured from $z = 0$, are ρ and ρ'. In the case of symmetry about the axis, ρ and ρ' are equal, and the coefficients of the terms of the third order vanish. The aberration then depends upon terms of the fourth order in x and y, and even these are made to vanish in the formulæ for the object-glasses of telescopes by the selection of suitable curvatures. In the theory of imperfectly constructed spectroscopes and of sloped lenses it is necessary to retain the terms of the third order, but we may assume a plane of symmetry $y = 0$, which is then spoken of as the *primary* plane. The equation of the wave-surface thus reduces to

$$z = \frac{x^2}{2\rho}+\frac{y^2}{2\rho'}+\alpha x^3+\gamma x y^2, \tag{2}$$

terms of higher order being omitted. In (2) ρ is the primary and ρ' the secondary focal length.

The equation of the normal at the point x, y, z is

$$z-\zeta = \frac{\xi-x}{x/\rho+3\alpha x^2+\gamma y^2} = \frac{\eta-y}{y/\rho'+2\gamma xy}, \tag{5}$$

and its intersection with the plane $\zeta=\rho$ occurs at the point determined approximately by

$$\xi = -\rho\,(3\alpha x^2+\gamma y^2), \qquad \eta = \frac{\rho'-\rho}{\rho'}y-2\rho\gamma xy, \tag{6}$$

terms of the third order being omitted.

According to geometrical optics, the thickness of the image of a luminous line (parallel to y) at the primary focus is determined by the extreme value of ξ, and for good definition it is necessary to reduce this thickness as much as possible. To this end it is necessary in general that both α and γ be small.

We will now examine more closely the character of the image at the primary focus in the case of a pencil originally of circular section. Unless $\rho'=\rho$, the second term in the value of η in (6) may be neglected. The rays proceeding from the circle $x^2+y^2=r^2$ intersect the plane $\zeta=\rho$ in the parabola

$$\frac{\rho\rho'^2(3\alpha-\gamma)}{(\rho'-\rho)^2}\eta^2-\xi = 3\alpha\rho r^2; \tag{7}$$

and the various parabolas corresponding to different values of r differ from one another only in being shifted along the axis of ξ. To find out how much of the parabolic arcs is included, we observe that for any given value of r the value of η is greatest when $x=0$. Hence the rays starting in the secondary plane give the remainder of the boundary of the image. Its equation, formed from (6) after putting $x=0$, is

$$\eta^2 = -\frac{(\rho'-\rho)^2}{\rho\rho'^2\gamma}\xi, \tag{8}$$

and represents a parabola touching the axis of η at the origin. The whole of the image is included between this parabola and the parabola of form (7) corresponding to the maximum value of r.

The width of the image when $\eta=0$ is $3\alpha\rho r^2$, and vanishes when $\alpha=0$, *i.e.*, when there is no aberration for rays in the primary plane. In this case the two parabolic boundaries coincide, and the image is reduced to a linear arc. If, further, $\gamma=0$, this arc becomes *straight*, and then the image of a short luminous line (parallel to y) is perfect to this order of approximation at the primary focus. In general, if $\gamma=0$, the parabola (8) reduces to the straight line $\xi=0$; that is to say, the rays which start in the secondary plane remain in that plane.

We will now consider the image formed at the secondary focus. Putting $\zeta = \rho'$ in (5), we obtain

$$\xi = \frac{\rho-\rho'}{\rho}x, \qquad \eta = -2\gamma\rho' xy. \tag{9}$$

If $\gamma = 0$, the secondary focal line is formed without aberration, but not otherwise. In general, the curve traced out by the rays for which $x^2+y^2 = r^2$, is

$$\left(\frac{\rho}{\rho-\rho'}\right)^2 \xi^2 + \frac{(\rho-\rho')^2\eta}{4\gamma^2\rho^2\rho'^2\xi^2} = r^2 \tag{10}$$

in the form of a figure of 8 symmetrical with respect to both axes. The rays starting either in the primary or in the secondary plane pass through the axis of ξ, the thickness of the image being due to the rays for which $x = y = r/\sqrt{2}$.*

Or if in order to find the intersection of the ray with the primary plane we put $\eta = 0$ in (5), we have approximately

$$\xi = \frac{(\rho-\rho')x}{\rho}, \qquad \zeta = \frac{1}{1/\rho'+2\gamma x},$$

showing that ζ is constant only when $\gamma = 0$.

The calculation of aberration for rays in the primary plane is carried out in the paper cited for the case of a thin lens sloped through a finite angle. If the curvature of the first surface be $1/r$ and of the second $1/s$, and if μ be the refractive index, the focal length f_1 in the primary plane is given by

$$\frac{1}{f_1} = \frac{\mu c'-c}{c^2}\left(\frac{1}{r}-\frac{1}{s}\right), \tag{11}$$

and the condition that there shall be no aberration is

$$\frac{(2\mu'+1)c}{u} + \frac{\mu'^2}{s} + \frac{\mu'-\mu'^2+1}{r} = 0. \tag{12}$$

Here u is the distance of the radiant point from the lens, ϕ the obliquity of the incident ray, ϕ' of the refracted ray, $c = \cos\phi$, $c' = \cos\phi'$, and $\mu' = \mu\cos\phi'/\cos\phi$.

A result, accordant with (12), but applicable only when ϕ is small, was given in another form by Mr. Dennis Taylor in 'Astron. Soc. Monthly Notices,' Ap., 1893.

If the incident rays be parallel, $u = \infty$, and the condition of freedom from aberration is

$$-\frac{r}{s} = \frac{1+\mu'-\mu'^2}{\mu'^2}. \tag{13}$$

* The above is taken from my "Investigations in Optics," 'Phil. Mag.,' 1879; 'Scientific Papers,' vol. 1, p. 441, and following. Some errata may be noted :—p. 441, line 9, insert y as factor in the first term of η; p. 443, line 9, for (7) read (8), line 10, for η read ξ.

As appears from (11), opposite signs for r and s indicate that both surfaces are convex.

If $\phi = 0$, $\mu' = \mu$, so that (13) gives, in this case,

$$-\frac{r}{s} = \frac{1+\mu-\mu^2}{\mu^2} \dots . \tag{14}$$

Thus, if $\mu = 1{\cdot}5$, the aberration vanishes for small obliquities when $s = -9r$. This means a double convex lens, the curvature of the hind surface being one-ninth of that of the front surface. If $s = \infty$, that is, if the lens be plano-convex with curvature turned towards the parallel rays,

$$1+\mu-\mu^2 = 0, \tag{15}$$

or

$$\mu = \tfrac{1}{2}(1+\sqrt{5}) = 1{\cdot}618.$$

Returning to finite obliquity, we see from (13) that whatever may be the index and obliquity of the lens, it is possible so to choose its form that the aberration shall vanish. If the form be plano-convex, the condition of no aberration is

$$1+\mu'-\mu'^2 = 0, \tag{16}$$

or

$$\mu' = \mu \cos\phi'/\cos\phi = 1{\cdot}618.$$

Here $\cos\phi' > \cos\phi$, and the ratio of the two cosines increases with obliquity from unity to infinity. Hence if $\mu > 1{\cdot}618$, there can be no freedom from aberration at any angle. When $\mu = 1{\cdot}618$, the aberration vanishes, as we have seen, when $\phi = 0$. If μ be less than $1{\cdot}618$, the aberration vanishes at some finite angle. For example, if $\mu = 1{\cdot}5$, this occurs when $\phi = 29°$.

In many cases the aberration of rays in the secondary plane is quite as important as that in the primary plane. In my former paper I gave a result applicable to a plano-convex lens, on the curved face of which parallel light falls. It was found that the secondary aberration vanished when the relation between obliquity and refractive index was such that

$$\sin^2\phi = \frac{3\mu^2-\mu^4-1}{3-\mu^2}. \tag{17}$$

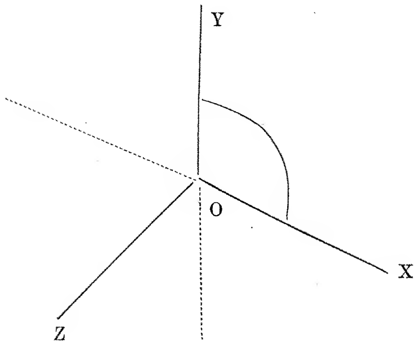
For small values of ϕ this gives the same index as before (15), inasmuch as

$$\mu^4-3\mu^2+1 = (\mu^2-1)^2-\mu^2 = (\mu^2-\mu-1)(\mu^2+\mu-1).$$

I inferred that for a plano-convex lens of index $1{\cdot}618$ neither kind of aberration is important at moderate slopes.

Having no note or recollection of the method by which (17) was obtained, and wishing to confirm and extend it, I have lately undertaken a fresh investigation, still limiting myself, however, to *parallel* incident rays. For simplicity, the lens may be supposed to come to a sharp circular edge, the

plane containing this edge being that of XY. The centre of the circle is the origin, and the axis of Z is the axis of the lens. The incident rays are parallel to the plane ZX, and make an angle Φ with OZ; so that Φ is the angle of incidence for the ray which meets the first surface of the lens at its central point. Everything is symmetrical with respect to the *primary* plane $y = 0$. It will suffice to consider the course of the rays which meet the lens close to its edge, of which the equation is $x^2+y^2 = \mathrm{R}^2$, if 2R be the diameter.



In order to carry out the calculation conveniently, we require general formulæ connecting the direction-cosines of the refracted ray with those of the incident ray and of the normal to the surface. If we take lengths AP, AQ along the incident and refracted rays proportional to μ, μ', the indices of the medium in which the rays travel, and drop perpendiculars PM, QN upon the normal MAN, then by the law of refraction the lines PM, NQ are equal and parallel; and the projection of PA+AM on any axis is equal to the projection of NA+AQ on the same axis. Thus if l, m, n are the direction-cosines of the incident ray, l', m', n' of the refracted ray, p, q, r of the normal taken in the direction from the medium in which the light is incident, ϕ, ϕ' the angles of incidence and refraction,

$$\mu l - \mu \cos\phi \,.\, p = -\mu' \cos\phi' \,.\, p + \mu' l'$$

and two similar equations. Hence

$$(\mu' l' - \mu l)/p = (\mu' m' - \mu m)/q = (\mu' n' - \mu n)/r = \mu' \cos\phi' - \mu \cos\phi. \quad (18)^*$$

Also $$\cos\phi = lp + mq + nr, \quad (19)$$

and ϕ' is given by $$\mu' \sin\phi' = \mu \sin\phi. \quad (20)$$

For our purpose there is no need to retain the two refractive indices, and for brevity we will suppose that the index outside the lens is unity and inside it equal to μ; so that in the above formulæ we are to write $\mu = 1$, $\mu' = \mu$. Hence

$$(\mu l' - l)/p = (\mu m' - m)/q = (\mu n' - n)/r = \mu \cos\phi' - \cos\phi. \quad (21)$$

Equation (19) remains as before, while (20) becomes

$$\mu \sin\phi' = \sin\phi. \quad (22)$$

* See Herman's 'Geometrical Optics,' Cambridge, 1900, p. 22.

For the first refraction at the point x, y, we have

$$l = \sin\Phi, \quad m = 0, \quad n = \cos\Phi;$$

and if χ_1 be the angle which the normal to the first surface at the edge of the lens makes with the axis,

$$p = \sin\chi_1 . x/\mathrm{R}, \qquad q = \sin\chi_1 . y/\mathrm{R}, \qquad r = \cos\chi_1;$$

so that

$$\frac{\mu l' - \sin\Phi}{x/\mathrm{R} . \sin\chi_1} = \frac{\mu m'}{y/\mathrm{R} . \sin\chi_1} = \frac{\mu n' - \cos\Phi}{\cos\chi_1} = \mu\cos\phi' - \cos\phi = \mathrm{C}_1, \tag{23}$$

and

$$\cos\phi = \sin\Phi\sin\chi_1 . x/\mathrm{R} + \cos\Phi\cos\chi_1. \tag{24}$$

In like manner if l'', m'', n'' be the direction-cosines of the twice refracted ray, p', q', r' those of the second normal, we may take

$$\frac{l'' - \mu l'}{x/\mathrm{R} . \sin\chi_2} = \frac{m'' - \mu m'}{y/\mathrm{R} . \sin\chi_2} = \frac{n'' - \mu n'}{\cos\chi_2} = \cos\psi' - \mu\cos\psi = \mathrm{C}_2, \tag{25}$$

if ψ, ψ' be respectively the angles of incidence and refraction at the second surface.

Here

$$\cos\psi = l'p' + m'q' + n'r'. \tag{26}$$

Eliminating l', m', n' between (23) and (25), we get

$$l'' = \sin\Phi + (\mathrm{C}_1\sin\chi_1 + \mathrm{C}_2\sin\chi_2)\, x/\mathrm{R},$$

$$m'' = (\mathrm{C}_1\sin\chi_1 + \mathrm{C}_2\sin\chi_2)\, y/\mathrm{R},$$

$$n'' = \cos\Phi + \mathrm{C}_1\cos\chi_1 + \mathrm{C}_2\cos\chi_2.$$

The equation of the ray after passage through the lens is

$$\frac{\xi - x}{l''} = \frac{\eta - y}{m''} = \frac{\zeta}{n''}. \tag{27}$$

The aberration in the secondary plane (depending on γ) is most simply investigated by inquiring where the ray (27) meets the primary plane $\eta = 0$. For the co-ordinates of the point of intersection,

$$\xi = x - \frac{l''y}{m''} = -\frac{\mathrm{R}\sin\Phi}{\mathrm{C}_1\sin\chi_1 + \mathrm{C}_2\sin\chi_2}, \tag{28}$$

$$\zeta = -\frac{n''y}{m''} = -\mathrm{R}\frac{\cos\Phi + \mathrm{C}_1\cos\chi_1 + \mathrm{C}_2\cos\chi_2}{\mathrm{C}_1\sin\chi_1 + \mathrm{C}_2\sin\chi_2}. \tag{29}$$

In interpreting (28), (29) we must remember that ζ is now measured parallel to the axis of the lens and not, as in the preliminary discussion, along the principal ray. Freedom from aberration requires that the line determined by varying x and y in (28), (29) should be perpendicular to the principal ray, or that $\zeta\cos\Phi + \xi\sin\Phi$ should be constant. And

$$-\frac{\zeta\cos\Phi + \xi\sin\Phi}{\mathrm{R}} = \frac{1 + (\mathrm{C}_1\cos\chi_1 + \mathrm{C}_2\cos\chi_2)\cos\Phi}{\mathrm{C}_1\sin\chi_1 + \mathrm{C}_2\sin\chi_2}. \tag{30}$$

Before proceeding further it may be well to compare (30) with known results when the aberration is neglected. For a first approximation we may identify ϕ and ϕ' with Φ and Φ', and also ψ and ψ' with Φ' and Φ respectively. Thus

$$C_1 = -C_2 = \mu \cos \Phi' - \cos \Phi. \tag{31}$$

Again, if r, s be the radii of the surfaces, we have, neglecting χ^2,

$$\chi_1 - \chi_2 = R/r - R/s; \tag{32}$$

and thence, from (30),

$$-\frac{1}{\zeta \cos \Phi + \xi \sin \Phi} = (\mu \cos \Phi' - \cos \Phi)\left(\frac{1}{r} - \frac{1}{s}\right), \tag{33}$$

the usual formula for the secondary focal length. The reckoning is such that the signs of r and s are opposite in the case of a doubly convex lens. We have now to proceed to a second approximation and inquire under what conditions (30) is independent of the particular ray chosen. In the numerator it is sufficient to retain the first power of χ_1, χ_2, so that we may take $\cos \chi_1$, $\cos \chi_2$ equal to unity; but in the denominator, which is already a small quantity of the first order, we must retain the terms of the second order in χ_1, χ_2. It is not necessary, however, to distinguish between the sines of χ_1, χ_2 and the angles themselves. The first step is to determine corrections to the approximate values of C_1 and C_2 expressed in (31).

For $\cos \phi$ itself we have, from (24),

$$\cos \phi = \cos \Phi + \chi_1 \, x/R \,.\, \sin \Phi;$$

and again

$$\mu \cos \phi' = \sqrt{\{\mu^2 - 1 + \cos^2 \phi\}} = \mu \cos \Phi' + \frac{\sin \Phi \cos \Phi}{\mu \cos \Phi'} \frac{\chi_1 x}{R},$$

so that

$$C_1 = (\mu \cos \Phi' - \cos \Phi)\left\{1 - \frac{\chi_1 x}{R} \frac{\sin \Phi}{\mu \cos \Phi'}\right\}. \tag{34}$$

In like manner, for C_2 in (26),

$$p' = \chi_2 x/R, \qquad q' = \chi_2 y/R, \qquad r' = 1,$$

so that

$$\mu \cos \psi = C_1 + \cos \Phi + \sin \Phi \,.\, \chi_2 x/R = \mu \cos \Phi' + \frac{x \sin \Phi}{R}\left(\frac{\chi_1 \cos \Phi}{\mu \cos \Phi'} - \chi_1 + \chi_2\right);$$

and

$$\cos \psi' = \sqrt{\{1 - \mu^2 + \mu^2 \cos^2 \psi\}}$$

$$= \cos \Phi + \frac{\mu \cos \Phi' \sin \Phi \,.\, x/R}{\cos \Phi}\left(\frac{\chi_1 \cos \Phi}{\mu \cos \Phi'} - \chi_1 + \chi_2\right);$$

whence

$$C_2 = (\cos \Phi - \mu \cos \Phi')\left\{1 - \frac{x \tan \Phi}{R}\left(\frac{\chi_1 \cos \Phi}{\mu \cos \Phi'} - \chi_1 + \chi_2\right)\right\}. \tag{35}$$

Thus, if we write $\mu' = \mu \cos \Phi' / \cos \Phi$,

$$C_1 + C_2 = \frac{x \sin \Phi}{R}(\mu' - 1)(\chi_2 - \chi_1); \tag{36}$$

and $\quad C_1\chi_1 + C_2\chi_2$

$$= \cos\Phi\,(\mu'-1)\,(\chi_1-\chi_2) + \frac{(\mu'-1)\,x\sin\Phi}{\mu'R}\{-\chi_1^2-(\mu'-1)\,\chi_1\chi_2+\mu'\chi_2^2\}, \tag{37}$$

in which $\quad \mu'\chi_2^2-(\mu'-1)\,\chi_1\chi_2-\chi_1^2 = (\chi_2-\chi_1)(\mu'\chi_2+\chi_1).$

Accordingly, $\quad \zeta\cos\Phi+\xi\sin\Phi$

$$= -\frac{R}{(\mu'-1)\cos\Phi\,(\chi_1-\chi_2)}\left\{1+\frac{x\tan\Phi}{\mu'R}\left[(\chi_2-\chi_1)(\mu'^2-\mu')\cos^2\Phi+\mu'\chi_2+\chi_1\right]\right\} \tag{38}$$

and the condition of no aberration is

$$(\chi_2-\chi_1)(\mu'^2-\mu')\cos^2\Phi+\mu'\chi_2+\chi_1 = 0. \tag{39}$$

Since χ_1, χ_2 are inversely proportional to r and s, we may write (39) in the form

$$\frac{1-(\mu'^2-\mu')\cos^2\Phi}{r}+\frac{\mu'+(\mu'^2-\mu')\cos^2\Phi}{s} = 0, \tag{40}$$

where
$$\mu' = \mu\cos\Phi'/\cos\Phi. \tag{41}$$

If $s = \infty$, so that the second surface is flat, we have as the special form of (40)

$$1-(\mu'^2-\mu')\cos^2\Phi = 0; \tag{42}$$

or in the case where $\Phi = 0$,

$$1+\mu-\mu^2 = 0, \tag{43}$$

the same condition as that (15) required to give zero aberration in the *primary* plane for small obliquities. In the case of finite obliquities we may write (42) in terms of μ,

$$\mu\cos\Phi\cos\Phi' = \mu^2\cos^2\Phi'-1, \tag{44}$$

or if we take the square of both sides of the equation,

$$\mu^2(1-\sin^2\Phi)(1-\sin^2\Phi') = (\mu^2\cos\Phi'-1)^2.$$

Of this the left-hand side may be equated to

$$(1-\sin^2\Phi)(\mu^2-\sin^2\Phi) = \mu^2-(\mu^2+1)\sin^2\Phi+\sin^4\Phi,$$

while on the right we have

$$(\mu^2-1-\sin^2\Phi)^2 = (\mu^2-1)^2-2(\mu^2-1)\sin^2\Phi+\sin^4\Phi;$$

so that
$$\sin^2\Phi = \frac{3\mu^2-\mu^4-1}{3-\mu^2}, \tag{45}$$

as formerly found (see (17)).

In interpreting (45), which we may also write in the form

$$\sin^2\Phi = \frac{(\mu^2-\mu-1)(\mu^2+\mu-1)}{\mu^2-3}, \tag{46}$$

we must bear in mind that it covers not only the necessary equation (44), but also the equation derived from (44) by changing the sign of one of the members. For instance, if we put $\mu = 1$ in (46), we derive $\sin^2\Phi = \frac{1}{2}$, or $\Phi = 45°$; but on referring back we see that these values satisfy, not (44), but

$$-\mu\cos\Phi\cos\Phi' = \mu^2\cos^2\Phi' - 1.$$

The transition occurs when $\cos\Phi = 0$, or $\Phi = 90°$, when (45) gives $\mu^2 = 2$, or $\mu = 1·4142$. For smaller values of μ there is no solution of (44). Onwards from this point, as μ increases, Φ diminishes. For example, when $\mu = 1·5$, $\sin^2\Phi = \frac{11}{12}$, whence $\Phi = 73°$. The diminution of Φ continues until $\mu^2-\mu-1 = 0$, or $\mu = 1·618$, when $\Phi = 0$, so that this is the value suitable for a plano-convex lens at small obliquities. After this value of μ is exceeded, $\sin^2\Phi$ in (46) is negative until $\mu^2 = 3$, or $\mu = 1·732$. When this point is passed, $\sin^2\Phi$ becomes positive, but a real value of Φ is not again reached. We infer that in the case of a plano-convex lens (curved face presented to parallel rays) there can be no freedom from secondary aberration unless μ lies between the rather narrow limits 1·414 and 1·618.

If the plano-convex lens be so turned as to present its plane face to the parallel rays, $r = \infty$; and (40) requires that

$$\mu' + (\mu'^2 - \mu')\cos^2\Phi = 0,$$

which cannot be satisfied, since $\mu' > 1$.

Leaving now the particular case of the plano-convex lens, let us suppose in the general formula (40) that $\Phi = 0$. We have

$$\frac{1+\mu-\mu^2}{r} + \frac{\mu^2}{s} = 0, \tag{47}$$

from which we see that, whatever may be the value of μ, compensation may be attained by a suitable choice of the ratio $r:s$. If $\mu < 1·618$, r and s have opposite signs, that is, the lens is double convex; while if $\mu > 1·618$, r and s have the same sign, or the lens is of the meniscus form. For example, if $\mu = 1·5$, (47) gives $s = -9r$, so that the lens is double convex, the hind surface having one-ninth the curvature of the front surface.

We have seen that the aberrations in both the primary and the secondary planes are eliminated for small obliquities in the case of a plano-convex lens if $\mu = 1·618$. The question arises whether this double elimination is possible at finite obliquities if we leave both the form of the lens and the refractive index arbitrary. It appears that this can *not* be done. The necessary condition is by (13), (40)

$$-\frac{s}{r} = \frac{\mu'^2}{1+\mu'-\mu'^2} = \frac{\mu'+(\mu'^2-\mu')\cos^2\Phi}{1-(\mu'^2-\mu')\cos^2\Phi},$$

or

$$\frac{\mu'}{1+\mu'-\mu'^2} = \frac{\mu'-(\mu'-1)\sin^2\Phi}{1+\mu'-\mu'^2+(\mu'^2-\mu')\sin^2\Phi},$$

whence

$$(\mu'^2-1)\sin^2\Phi = 0, \tag{48}$$

which can be satisfied only by $\Phi = 0$, since $\mu' > 1$.

Since it is not possible to destroy both the primary and secondary aberrations when the angle of incidence is finite, it only remains to consider a little further in detail one or two special cases.

We have already spoken of the plano-convex lens; but for a more detailed calculation it may be well to form the equation for absence of primary aberration analogous to (45). From (16),

$$\mu\cos\phi'\cos\phi = \mu^2-1, \tag{49}$$

whence, if we square both sides,

$$\sin^4\phi-(\mu^2+1)\sin^2\phi+3\mu^2-\mu^4-1 = 0,$$

giving

$$\sin^2\phi = \frac{\mu^2+1\pm\sqrt{5}\,(\mu^2-1)}{2}, \tag{50}$$

so that

$$\sin^2\phi = 1{\cdot}618034-0{\cdot}618034\,\mu^2, \tag{51}$$

the other root being excluded if $\mu > 1$. It may be remarked that there is no distinction between ϕ here and Φ in (45).

The following table will give an idea of the values of ϕ from (51) and (45) for which the plano-convex lens of variable index is free from aberration in the primary and secondary planes respectively.

μ.	ϕ. Primary plane.	ϕ. Secondary plane.
	° ′	° ′
1·0000	90 0	—
1·4142	38 11	90 0
1·5000	28 29	73 13
1·5500	21 24	58 37
1·5900	13 38	39 45
1·6000	10 55	32 25
1·6100	7 16	22 1
1·6180	0 0	0 0

In the above the curved face is supposed to be presented to the parallel rays. If the lens be turned the other way, $r = \infty$, and (13) gives $\mu' = 0$, an equation which cannot be satisfied. In this case neither the primary nor the secondary aberration can be destroyed at any angle.

Next suppose that the lens is equi-convex, so that $s = -r$. In this case (13) gives

$$\mu'^2 - \tfrac{1}{2}\mu' - \tfrac{1}{2} = 0, \tag{52}$$

whence $\mu' = 1$, or $-\frac{1}{2}$, of which the latter has no significance. Also from (40) we get $\mu' = 1$. It appears that neither aberration can vanish for an equi-convex lens, unless in the extreme case $\mu = 1$, $\phi = 0$, when the lens produces no effect at all.

Part II.

It is a common experience in optical work to find the illumination deficient when an otherwise desirable magnification is introduced. Sometimes there is no remedy except to augment the intensity of the original source of light, if this be possible. But in other cases the defect may largely depend upon the manner in which the magnification is effected. With the usual arrangements magnifying takes place equally in the two perpendicular directions, though perhaps it may only be required in one direction. For example, in observations upon the spectrum, or upon interference bands, there is often no need to magnify much, or perhaps at all, in the direction parallel to the lines or bands. If, nevertheless, we magnify equally in both directions, there may be an unnecessary and often very serious loss of light.

In discussing this matter there is another distinction to be borne in mind. Sometimes it is not necessary or advantageous that there should exist a point-to-point correspondence between the object and the image. It suffices that a point in the object be represented in the image by a narrow line. This happens, for example, in the use of Rowland's concave gratings. A conspicuous instance occurs in the refractometer which I described in connection with observations upon argon and helium.* Here while the object-glass of the telescope was as usual, a very high magnification in one direction was secured by the use, as sole eye-piece, of a cylindrical lens taking the form of a glass rod 4 mm. in diameter. An equal magnification in both directions, such as would have been afforded by the usual spherical eye-pieces, would have so reduced the light as to make the observations impossible.

Whenever the field of view varies only in one dimension, there is usually no loss, and there may even be gain in the presence of astigmatism. In other cases a point-to-point correspondence between image and object is desirable or necessary, and the question arises how it may best be attained otherwise than by the use of a common telescope, which limits the mag-

* 'Roy. Soc. Proc.,' vol. 59, p. 198, 1896; 'Scientific Papers,' vol. 4, pp. 218, 364.

nification in the two directions to equality. I had occasion to consider this problem in connection with observations upon Haidinger's rings as observed with a Fabry and Perot apparatus. Here the field is symmetrical about an axis, and all the advantage that magnification can give is secured though it take place only in one direction. At the same time light is usually saved by abstaining from magnifying in the second direction also. In this way the circular rings assume an elongated elliptical form—a transformation which in no way prejudices observation by simple inspection. The question whether light is saved, as compared with symmetrical magnification, depends of course upon the aperture available in the two directions. In a Fabry and Perot apparatus this is usually somewhat restricted.

One simple solution of the problem, available when the light is homogeneous, may be found in the use of a *magnifying prism*, that is a prism so held that the emergence is more nearly grazing than the incidence. In this way we may obtain a moderate magnification in one direction combined with none at all in the second direction. A magnification equal in both directions may then be superposed with the aid of a common telescope. This method would probably answer well in certain cases, but it has its limitations. Moreover, the accompanying deviation of the rays through a large angle would often be inconvenient.

If we are allowed the use of cylindrical lenses, or of lenses whose curvature though finite is different in the two planes, we may attain our object with a construction analogous to that of a common telescope. Suppose that the eye-piece is constituted of a spherical and a contiguous cylindrical convex lens. In one plane the power of the eye-piece is greater than in the other perpendicular plane. Thus, if the object-glass be composed of spherical lenses only, there cannot be complete focussing. With the spherical lens or lenses of the object-glass, mounted as usual, it is necessary to combine a cylindrical lens of comparatively feeble power, which may be either convex or concave. All that is necessary to constitute a telescope in the full sense of the word, that is an apparatus capable of converting incident parallel rays into emergent parallel rays, is that the usual condition connecting the focal lengths of object-glass and eye-piece should be satisfied for the two principal planes taken separately. The magnifying powers in the two planes may thus be chosen at pleasure; and since there is symmetry with respect to both planes the apparatus is free from the unsymmetrical aberration expressed in (1).

When the magnifying desired is considerable in both planes, there is but little for the cylindrical component of the object-glass to do, and it occurred to me that it might be dispensed with, provided a moderate slope were given

to the single (spherical) lens. In the earlier experiments the object-glass was a nearly equi-convex lens of 14 inches focus. The eye-piece was a combination of a spherical lens of 6 inches focus with a cylindrical lens of $2\frac{1}{2}$ inches focus, so that the focal lengths of the combination were about 2 inches and 6 inches in the principal planes, giving a *ratio* of magnifications as three to one. With the above object-lens the actual magnifications would be about 2 and 6. During the observations the axis of the telescope was horizontal and that of the cylindrical lens vertical, so that the higher magnification was in the horizontal direction of the field. During the adjustments it is convenient to examine a cross formed by horizontal and vertical lines, ruled upon paper well illuminated and placed at a sufficient distance.

When the object-lens stands square, there is, of course, no position of the compound eye-piece which allows both constituents of the cross to be seen in focus together. If we wish to pass from the focus for the horizontal to that necessary for the vertical line, we must push the eye-piece in. In order to focus both at once we must slope the object-lens. And since while both the primary and secondary focal lengths are diminished by obliquity the former is the *more* diminished, it follows that the sloping required is in the vertical plane, the lens being rotated about its horizontal diameter. If we introduce obliquity by stages, we find that the displacement of the eye-piece required to pass from one focus to the other gradually diminishes until an obliquity is reached which allows both lines of the cross to be in focus simultaneously. At a still higher obliquity the relative situation of the two foci is reversed. In the actual experiment with the 14-inch object-lens, the critical obliquity was roughly estimated at about 30°.

The above apparatus worked fairly well when tried upon interference rings from a thallium vacuum tube. But it was evident that the image suffered somewhat from aberration. A better result ensued when the magnification in both directions was increased by the substitution of an object-lens of 24 inches focus, although this also was equi-convex.

Being desirous of testing the method of the sloped lens under more favourable conditions, I procured from Messrs. Watson a lens of baryta crown glass of index for mean rays 1·59, and of *plano*-convex form. The aperture was about $1\frac{3}{4}$ inch, and the focal length 24 inches. When this was combined with the compound eye-piece already described, the performance was very good, if, in accordance with the indications of theory, the curved face of the object-lens was presented to the incident light. The test may be made either upon a cross or upon a system of concentric circles drawn upon paper. The angle of slope giving the best effect was now very sharply

defined. When, however, the object-lens was reversed, so as to present its plane face to the incident rays, no good result could be attained, evidently in consequence of aberration. The change in the character of the image was now very apparent when the eye was moved up and down, the rings appearing more elliptical as the eye moved in the direction of the nearest part of the edge of the sloped lens. Next to nothing of this effect could be observed when the object-lens was used in the proper position. It is scarcely necessary to say that care must be taken to ensure that the axis, about which the lens is turned, is truly perpendicular to the axis of the cylindrical component of the eye-piece.

Altogether it appears that the combination of sloped object-lens with compound cylindrical eye-piece constitutes a satisfactory solution of the problem. I believe that it may be applied with advantage in the many cases which arise in the laboratory where an unsymmetrical magnifying best meets the conditions. The question as to the precise index to be chosen for the plano-convex lens remains to some extent open. Possibly a somewhat higher index, *e.g.*, 1·60, or even 1·61, might be preferred to that which I have used.

With the view to the design of future instruments, it may be convenient to set out the formula giving the distance between the primary and secondary foci of the object-lens as dependent upon the obliquity ϕ. If f_1, f_2 be the primary and secondary focal lengths, it is known (compare (33)) that

$$\frac{f_0}{f_1} = \frac{\mu \cos \phi' - \cos \phi}{(\mu - 1) \cos^2 \phi}, \tag{53}$$

$$\frac{f_0}{f_2} = \frac{\mu \cos \phi' - \cos \phi}{\mu - 1}, \tag{54}$$

f_0 being the focal length corresponding to $\phi = 0$; so that

$$\frac{f_2 - f_1}{f_0} = \frac{(\mu - 1) \sin^2 \phi}{\mu \cos \phi' - \cos \phi}. \tag{55}$$

In this

$$\mu \cos \phi' - \cos \phi = \sqrt{(\mu^2 - \sin^2 \phi)} - \sqrt{(1 - \sin^2 \phi)} = (\mu - 1)\left\{1 + \frac{\sin^2 \phi}{2\mu}\right\}$$

approximately. Hence

$$\frac{f_2 - f_1}{f_0} = \sin^2 \phi \left\{1 - \frac{\sin^2 \phi}{2\mu}\right\}, \tag{56}$$

from which the required obliquity is readily calculated when the nature of the eye-piece and the focal length of the object-lens are given.

P.S., June 6.—From von Rohr's excellent 'Theorie und Geschichte des Photographischen Objectivs,' Berlin, 1899, I learn that Rudolf and, at a still

earlier date (1884), Lippich had proposed a different method of obtaining a diverse magnification, and one that I had overlooked. This consists in the use of an eye-piece formed by crossing two cylindrical lenses of different powers. The two lenses are mounted, not close together, but at such distances from the image as to render parallel the rays diverging from it in the two planes separately. In this method the object-lens remains square to the axis of the instrument. Lippich had the same object in view as that which guided me. I have tried his method with success, obtaining an image as good, or nearly as good, as that afforded by the sloped lens. I understand that Professor S. P. Thompson also has used a similar device.

The Optical Constants of Gypsum at different Temperatures, and the Mitscherlich Experiment.

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At a lecture delivered to the Prussian Academy of Sciences in the year 1826, Professor Mitscherlich showed an optical experiment with gypsum (selenite) which has ever since been known as the "Mitscherlich experiment." He had discovered, as the most striking result of an investigation of the double refraction of a number of crystallised substances at varying temperatures, that gypsum, the crystallised hydrated sulphate of lime, $CaSO_4.2H_2O$, suffers greater change, as regards the position of its optic axes, under the influence of heat than any other substance then examined. At the ordinary temperature it is biaxial, with an optic axial angle of about 60°, but on raising the temperature the angle diminishes until in the neighbourhood of the temperature of boiling water the axes come together, producing in convergent polarised light the rectangular cross and circular rings of a uniaxial crystal. Beyond that temperature the axes again separate, but in the direction at right angles to their former one. On allowing the crystal section-plate to cool, the phenomena are repeated in the reverse order. It was this striking experiment which was shown for the first time by Mitscherlich in the lecture in Berlin above referred to.

The experiment is one that is often now repeated, as one of the most interesting and easily demonstrated cases of "crossed-axial-plane dispersion." Other well-known cases are brookite, the rhombic form of titanium dioxide